# THE PERIODICITY OF NIM-SEQUENCES IN TWO-ELEMENT SUBTRACTION GAMES[1]

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**Abstract**

Consider a subtraction game with a two-element subtraction set $\{x, y\}$. We offer a new proof of an old result that states that the periodicity of the nim-values for the game is either $2x$ or $x + y$. Moreover, we completely classify the behavior of the game by describing necessary and sufficient conditions for obtaining a given nim-value at a position.

[1] In memory of Elwyn Berlekamp, John Conway, and Richard Guy

## 1. Introduction

Consider the following two-player game. The game begins with a pile of stones between the two players. The players take turns removing stones from the pile, under the restriction that the number of stones one removes in a turn is in a predetermined set $S$ of natural numbers called a *subtraction set*. The player who takes the last stone wins. Such games are called subtraction games. Subtraction games are widely studied (e.g. [1, 2, 3, 4]) and are related to the game of Nim. In particular, the famous Sprague–Grundy Theorem [2, p. 56] implies that any subtraction game is equivalent to a game of Nim having only a single pile; the size of such a pile is said to be the *nim-value*, or Grundy number, of a subtraction game.

Given a subtraction set, we can build a so-called *nim-sequence* of nim-values $\mathcal{G}(0), \mathcal{G}(1), \ldots,$ where the nim-value of a game with $m$ stones is $\mathcal{G}(m)$. Problem **A1(1)** of "Unsolved Problems in Combinatorial Games" [5] asks to analyze the nim-sequences of subtraction games, and particularly asks about the periodicity of the nim-sequences.

In this paper, we focus on the periodicity of a game with subtraction set $\{x, y\}$ where $x < y$. This question has previously been answered in [1, p. 97] and [4, pp. 529–530] via the following theorem, where the notation $(a_1a_2\ldots a_n)^k$ denotes a repetition of $a_1a_2\ldots a_n$ a total of $k$ times, and $(a_1a_2\ldots a_n)$ denotes $(a_1a_2\ldots a_n)^\infty$.

**Theorem 1 ([1, 4]).** *Let $S = \{x, y\}$ with $x < y$. Then the nim-sequence for this subtraction game is*

- $(0^x1^x)$ *if $y = (2m+1)x$ for some integer $m$;*
- $((0^x1^x)^m0^r2^{x-r}1^r)$ *if $y = 2xm + r$ for some integers $m$ and $r$ with $0 \le r < x$;*
- $((0^x1^x)^m2^{x+r})$ *if $y = 2xm + r$ for some integers $m$ and $r$ with $-x < r < 0$.*

This paper offers a new proof of this result by using a result from Ferguson [2, p. 86] with two new results.

**Theorem 2. ([2, Ferguson's Pairing Property])** *Let $S = \{s_1, s_2, s_3, \ldots, s_k\}$ with $s_1 < s_i$ for all $i > 1$. Then $\mathcal{G}(n) = 1$ if and only if $\mathcal{G}(n - s_1) = 0$.*

**Theorem 3.** *Let $S = \{x, y\}$ with $x < y$. Then $\mathcal{G}(n) = 2$ if and only if both*

1. *$\mathcal{G}(n - x) = 1$ and*
2. *$\mathcal{G}(n - y) = 0$.*

**Theorem 4.** *Let $S = \{x, y\}$ with $x < y$. If $n \ge y$, then $\mathcal{G}(n) = 0$ if and only if $\mathcal{G}(n - y) = 1$.*

It is not difficult to observe that the nim-value of this subtraction game must be in $\{0, 1, 2\}$, so these three theorems give a complete characterization of this type of subtraction game.

Note that there was no proof of Theorem 1 in [4]. Additionally, the proof of Theorem 1 in [1] uses a method that does not give the same detailed information about the structure of this particular subtraction game. Thus, the new proof of Theorem 1 from Theorems 2, 3, and 4 gives additional insight into the behavior of this game.

While the results originally done in [1, p. 97] and [4, pp. 529–530] completely classify the periodicity of the nim-values of the game for a subtraction set of size 2, this work has not yet been done for a subtraction set of size 3.

## 2. Preliminaries

We will denote $\{0, 1, 2, 3, \ldots\}$ by $\mathbb{N}$ and consider the game described in the introduction. We define the following.

**Definition 1.** For a given set $A = \{x_1, \ldots, x_m\} \subseteq \mathbb{N}$, the *minimum excluded value* of $A$ is the smallest non-negative integer $x$ such that $x \notin A$. We denote the minimum excluded value of $A$ as $\operatorname{mex}(A)$.

**Definition 2.** For a given subtraction set $S = \{s_1, \ldots, s_k\}$, the *nim-value* of a non-negative integer $n$ is defined to be $\operatorname{mex}\{\mathcal{G}(n-s) \mid s \in S, n-s \geq 0\}$. We will denote the nim-value of $n$ as $\mathcal{G}(n)$.

Note that if $n < \min S$, then $\mathcal{G}(n) = \operatorname{mex} \emptyset = 0$. We now state and prove an elementary result alluded to in the Introduction.

**Proposition 1.** *Let $S = \{s_1, \ldots, s_k\}$. Then $\mathcal{G}(n) \leq k$ for all $n$.*

*Proof.* Fix $n \in \mathbb{N}$. Then $\mathcal{G}(n) = \operatorname{mex} T$, where $T$ is defined to be $\{\mathcal{G}(n-s) \mid s \in S, n-s \geq 0\}$. From the way $T$ is defined, it is clear that $|T| \leq |S| = k$, so $\mathcal{G}(n)$ is the smallest number missing from a set of at most $k$ elements. Thus, one of the $k+1$ values in $\{0, 1, 2, \ldots, k\}$ must be missing from $T$, and we conclude that $\mathcal{G}(n) = \operatorname{mex} T \leq k$. □

The next definition will be convenient in two proofs.

**Definition 3.** Let $x, m, n \in \mathbb{N}$ such that $x \geq 2$. We say that $m$ is *$x$-even* if $m \equiv k \pmod{2x}$ for some $k \in \{0, 1, \ldots, x-1\}$, and we say that $m$ is *$x$-odd* otherwise. We say that $m$ and $n$ have the same *$x$-parity* if $m$ and $n$ are both $x$-even or both $x$-odd.

**Proposition 2.** *If the subtraction set is $\{x\}$, then $\mathcal{G}(n) = 0$ if $n$ is $x$-even and* 1 *otherwise. Thus, the nim-sequence is $(0^x 1^x)$ and the period is $2x$.*

*Proof.* We will prove this result by induction on $n$. If $n < x$, then $\mathcal{G}(n) = \text{mex}\,\emptyset = 0$. Now suppose that $n \geq x$ and the result holds for values smaller than $n$. Then $\mathcal{G}(n) = \text{mex}\{\mathcal{G}(n-x)\}$. If $n$ is $x$-even, $n - x$ must be $x$-odd, and we conclude by induction that $\mathcal{G}(n) = \text{mex}\{1\} = 0$. Similarly, if $n$ is $x$-odd, then $n - x$ is $x$-even and $\mathcal{G}(n) = \text{mex}\{0\} = 1$. Thus, nim-sequence is $(0^x 1^x)$ and the period is $2x$. □

## 3. The Proofs of the Ferguson-like Theorems

Theorem 2 (Ferguson's Pairing Property) is a general result that works for subtraction sets of any size. The proof can be found at [2, p. 86] and is similar to the proof of Theorem 4.

Our strategy for determining the periodicity of the nim-values for a subtraction $\{x, y\}$ will be to determine Ferguson-like rules for determining exactly when $\mathcal{G}(n) = 0$ and $\mathcal{G}(n) = 2$. We start with the criteria for when $\mathcal{G}(n) = 2$ since we will use the result in the proof of Theorem 4.

**Theorem 3.** Let $S = \{x, y\}$ with $x < y$. Then $\mathcal{G}(n) = 2$ if and only if both

1. $\mathcal{G}(n-x) = 1$ and
2. $\mathcal{G}(n-y) = 0$.

*Proof.* First, suppose that $\mathcal{G}(n-x) = 1$ and $\mathcal{G}(n-y) = 0$. Then $\mathcal{G}(n) = \text{mex}\{\mathcal{G}(n-x), \mathcal{G}(n-y)\} = \text{mex}\{1, 0\} = 2$. So suppose that $\mathcal{G}(n) = 2$. Since $2 = \mathcal{G}(n) = \text{mex}\{\mathcal{G}(n-x), \mathcal{G}(n-y)\}$, we conclude that $\{\mathcal{G}(n-x), \mathcal{G}(n-y)\} = \{0, 1\}$. Suppose toward a contradiction that $\mathcal{G}(n-x) = 0$. Then by Theorem 2, we conclude that $1 = \mathcal{G}(n) = 2$, a contradiction. Therefore, $\mathcal{G}(n-x) = 1$ and $\mathcal{G}(n-y) = 0$. □

We can now determine the final Ferguson-like property.

**Theorem 4.** Let $S = \{x, y\}$ with $x < y$. If $n \geq y$, then $\mathcal{G}(n) = 0$ if and only if $\mathcal{G}(n-y) = 1$.

*Proof.* Let $n$ be the smallest value for which this fails, and suppose first that $\mathcal{G}(n) \neq 0$ and $\mathcal{G}(n-y) = 1$. Then $0 \neq \mathcal{G}(n) = \text{mex}\{\mathcal{G}(n-x), \mathcal{G}(n-y)\}$. Since $\mathcal{G}(n-y) = 1$, we have that $\mathcal{G}(n) \neq 1$ and determine that $\mathcal{G}(n) = 2$ by Proposition 1. By Theorem 3, we conclude that $0 = \mathcal{G}(n-y) = 1$, a contradiction.

Now suppose that $\mathcal{G}(n) = 0$ and $\mathcal{G}(n-y) \neq 1$. Then $0 = \mathcal{G}(n) = \text{mex}\{\mathcal{G}(n-x), \mathcal{G}(n-y)\}$, so $\mathcal{G}(n-y) \neq 0$. Thus, $\mathcal{G}(n-y) = 2$ by Proposition 1. By Theorem 3, we conclude that $\mathcal{G}((n-x)-y) = \mathcal{G}((n-y)-x) = 1$. Since $n$ is the minimal counterexample, $n - x < n$, and $\mathcal{G}((n-x)-y) = 1$, we conclude that $\mathcal{G}(n-x) = 0$. Then $0 = \mathcal{G}(n) = \text{mex}\{\mathcal{G}(n-x), \mathcal{G}(n-y)\} = \text{mex}\{0, 2\} = 1$, a contradiction. □

## 4. The Period of a 2-element Subtraction Game

We can now bound the period when the subtraction set is $\{x, y\}$.

**Proposition 3.** *Let $S = \{x, y\}$ with $x < y$. Then the nim-values are periodic with period at most $x + y$.*

*Proof.* It is sufficient to prove that $\mathcal{G}(n - x - y) = \mathcal{G}(n)$ for all $n \geq x + y$. So fix an $n \geq x + y$. By Proposition 1, $\mathcal{G}(n) \in \{0, 1, 2\}$. If $\mathcal{G}(n) = 0$, then $\mathcal{G}(n - y) = 1$ by Theorem 4, and $\mathcal{G}(n - x - y) = \mathcal{G}((n - y) - x) = 0$ by Theorem 2. If $\mathcal{G}(n) = 1$, then $\mathcal{G}(n - x) = 0$ by Theorem 2, and $\mathcal{G}(n - x - y) = \mathcal{G}((n - x) - y) = 1$ by Theorem 4.

Finally, suppose that $\mathcal{G}(n) = 2$. Then $\mathcal{G}(n - x) = 1$ and $\mathcal{G}(n - y) = 0$ by Theorem 3. Since $1 = \mathcal{G}(n - x) = \text{mex}\{\mathcal{G}(n - 2x), \mathcal{G}(n - x - y)\}$ and $0 = \mathcal{G}(n - y) = \text{mex}\{\mathcal{G}(n - 2y), \mathcal{G}(n - x - y)\}$, we conclude that $\mathcal{G}(n - x - y)$ can be neither 0 nor 1. By Proposition 1, we conclude that $\mathcal{G}(n - x - y) = 2$. Thus, $\mathcal{G}(n) = \mathcal{G}(n - x - y)$ holds in every possible value of $\mathcal{G}(n)$. □

We now prove Theorem 1 with two separate results.

**Proposition 4.** *If the subtraction set is $\{x, (2m + 1)x\}$ for some $m \geq 1$, then $\mathcal{G}(n) = 0$ if $n$ is $x$-even and $\mathcal{G}(n) = 1$ if $n$ is $x$-odd. In particular, the nim-sequence is $(0^x 1^x)$ and the period is $2x$.*

*Proof.* We induct on $n$. The result holds for $n < (2m + 1)x$ by Proposition 2. So suppose $n \geq (2m + 1)x$ and that the result holds for values smaller than $n$. Then $\mathcal{G}(n) = \text{mex}\{\mathcal{G}(n - x), \mathcal{G}(n - (2m + 1)x)\}$. Since $n - x$ and $n - (2m + 1)x$ both have different $x$-parity than $n$, we conclude that $\mathcal{G}(n) = \text{mex}\{0\} = 1$ if $n$ is $x$-odd and $\mathcal{G}(n) = \text{mex}\{1\} = 0$ if $n$ is $x$-even by induction. Thus, the nim-sequence is $(0^x 1^x)$ and the period is $2x$. □

Every other case is handled in the following proposition.

**Proposition 5.** *Let $S = \{x, y\}$ such that $x < y$ and $y$ is not an odd multiple of $x$. Then the period of the subtraction game with subtraction set $S$ is $x + y$.*

*Proof.* By Proposition 3, it is sufficient to show that the sequence is not periodic within the first $x + y$ terms. We write $y = (2x)m + r$ for some $m$ and $r$ such that $-x < r < x$. An easy calculation shows that the first $x + y$ terms of the sequence are

$$(0^x 1^x)^m 0^r 2^{x-r} 1^r$$

if $0 \leq r < x$ and

$$(0^x 1^x)^m 2^{x+r}$$

otherwise. In either case, the period is not less than $x + y$ by inspection, so we conclude by Proposition 3 that the period is $x + y$ exactly. □

## References

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